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ORIGINAL PAPER

# A Bernstein property of solutions to a class of prescribed affine mean curvature equations

**James Alexander McCoy**

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**Abstract** Let $x\colon M \to A^{n+1}$ be a locally strongly convex hypersurface, given as the graph of a locally strongly convex function $x_{n+1} = z(x_1, \ldots, x_n)$. In this paper we prove a Bernstein property for hypersurfaces which are complete with respect to the metric $G^\sharp = \sum \left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right) dx_i dx_j$ and which satisfy a certain Monge–Ampère type equation. This generalises in some sense the earlier result of Li and Jia for affine maximal hypersurfaces of dimension $n = 2$ and $n = 3$ (Li, A.-M., Jia, F.: A Bernstein property of affine maximal hypersurfaces. Ann. Glob. Anal. Geom. **23**, 359–372 (2003)), related results (Li, A.-M., Jia, F.: Locally strongly convex hypersurfaces with constant affine mean curvature. Diff. Geom. Appl. **22**(2), 199–214 (2005)) and results for $n = 2$ of Trudinger and Wang (Trudinger, N.S., Wang, X.-J.: Bernstein-Jörgens theorem for a fourth order partial differential equation. J. Partial Diff. Equ. **15**(2), 78–88 (2002)).

**Keywords** Bernstein property · Monge–Ampère equation · Affine mean curvature · Entire graphs

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## 1 Introduction

Let $x\colon M \to A^{n+1}$ be a hypersurface given by a locally strongly convex function

$$x_{n+1} = z(x_1, \ldots, x_n)$$

J. A. McCoy (✉)
School of Mathematics and Applied Statistics,
University of Wollongong,
Wollongong,
NSW 2420, Australia
e-mail: jamesm@uow.edu.au

defined in a domain $\Omega \subset A^n$. As in [11], there is no loss of generality in introducing a Euclidean structure in $A^{n+1}$ so we may work directly in $\mathbb{R}^{n+1}$.

By *locally strongly convex* we mean that the Hessian of $z$ is positive definite. Let

$$\rho\,(x_1,\ldots,x_n) = \left[\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)\right]^{-\frac{1}{(n+2)}} (x_1,\ldots,x_n)\,. \tag{1}$$

We consider hypersurfaces $x\,(M)$ which are complete with respect to the Schwarz–Pick metric [4]

$$G^\sharp = \sum \left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right) dx_i dx_j.$$

A related metric is the Berwald–Blaschke metric given by

$$G = \rho\, G^\sharp. \tag{2}$$

It is a result from affine differential geometry that the affine mean curvature $L_1$ of the hypersurface $x\,(M)$ (i.e. mean curvature in the Blaschke metric) is given by

$$L_1 = -\frac{1}{n\rho}\Delta\rho. \tag{3}$$

We refer the reader to [8] for details of this. Here $\Delta$, the Laplacian with respect to the Blaschke metric, is given by

$$\Delta = \frac{1}{\sqrt{\det G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(G^{ij}\sqrt{\det G}\frac{\partial}{\partial x_j}\right),$$

where $\left(G^{ij}\right) = \left(G_{ij}\right)^{-1}$.

In this paper we look at hypersurfaces $x\,(M)$ which satisfy the following fourth order fully nonlinear Monge–Ampère type equation:

$$\Delta\rho = \frac{\eta}{\rho}\,\|\nabla\rho\|_G^2 - f\,(\rho) \tag{4}$$

for certain constants $\eta$ and certain differentiable functions $f\colon \mathbb{R}^+ \to \mathbb{R}$. In view of (3), asking that $x\,(M)$ satisfies this PDE is equivalent to prescribing its affine mean curvature, namely

$$L_1 = -\frac{\eta}{n\rho^2}\,\|\nabla\rho\|_G^2 + \frac{f\,(\rho)}{n\rho}. \tag{5}$$

Note also that when $f \equiv 0$, (4) is equivalent to

$$\Delta\rho^{1-\eta} = 0. \tag{6}$$

In this case, if we make the conformal change of metric

$$\tilde{G} = \rho^{-\frac{2\eta}{n-2}} G$$

then (6) may be rewritten as

$$\tilde{\Delta}\rho = 0,$$

where $\tilde{\Delta}$ is the Laplacian with respect to the $\tilde{G}$ metric. This PDE is the Euler Lagrange equation associated with extremising the energy functional

$$F\,[z] = \int \rho^{\eta-1} dx.$$

We rewrite the PDE in terms of coordinate derivatives in Remark 1 below.

Our result is the following Bernstein-type property:

**Theorem 1.1** *Let x, z and $\rho$ be defined as above and suppose $x(M)$ is complete with respect to the metric $G^\sharp$. If also $\rho$ satisfies Eq. 4 where the constant $\eta$ satisfies*

$$\eta > \frac{1}{2}\left\{n + \frac{(n+2)(n-1)}{2\sqrt{n}}\right\} \text{ or } \eta < \frac{1}{2}\left\{n - \frac{(n+2)(n-1)}{2\sqrt{n}}\right\}, \tag{7}$$

*and the differentiable function f satisfies $f(C) = 0$ for some $C > 0$ and*

$$f' - \left(\frac{n-2\eta}{n-1}\right)\frac{f}{\rho} \leq 0, \tag{8}$$

*then $x(M)$ is an elliptic paraboloid.*

**Remarks**

1. In [12], Trudinger and Wang prove, using PDE methods, the Bernstein property for 2-dimensional surfaces given as graphs of uniformly convex $z$ satisfying Eq. 6, where $\eta < -3$. Crucial to their techniques are upper and lower a priori bounds on $\rho$ and the observation that (6) can be rewritten as

$$Z^{ij}D_{ij}\rho^{\eta+n+1} = 0,$$

where $\left(Z^{ij}\right)$ is the cofactor matrix of the Hessian of $z$. This form is appropriate for applying the Cafarelli–Gutièrrez Hölder estimate [3] and Caffarelli Schauder estimate [2] for regularity of solutions to Monge–Ampère equations. In this paper we enlarge the set of allowable $\eta$, allow $n \geq 2$ and allow certain $f$ terms as in the statement of the theorem, with the condition that the hypersurface $x(M)$ is complete with respect to $G^\sharp$.
2. If $n = 2$ or $n = 3$ then the condition (7) allows $\eta = 0$. For this value of $\eta$ and when $f \equiv 0$, (6) is the affine maximal hypersurface equation and we recover from Theorem 1.1 the Bernstein property of 2- and 3-dimensional affine maximal hypersurfaces proved by Li and Jia in [6].
3. Types of $f$ which satisfy the conditions of the theorem include
   - $f(\rho) = A\rho^p + B$, where $A$ and $B$ are of opposite sign for the root condition, and $p$ is chosen such that (8) holds given $\eta$ satisfying (7).
   - If we set
   $$f' - \left(\frac{n-2\eta}{n-1}\right)\frac{f}{\rho} = -K$$
   say, for some $K > 0$, then we get solutions
   $$f(\rho) = \frac{(n-1)K}{(1-2\eta)}\rho + C_0\rho^{\left(\frac{n-2\eta}{n-1}\right)}.$$
   Depending on $\eta$ we can choose the sign of the constant $C_0$ such that $f$ has a positive root.
   - Other interesting $f$'s constructed piecewise.
4. The requirement that $f$ has a root at some positive $\rho$ comes into the proof of Theorem 1.1 when we show that provided (8) holds, the solution $\rho$ to (4) must be identically constant. Therefore, for (4) itself to be satisfied, we must choose this

constant to be a root of $f$. If $f$ satisfies (8) but does not have a zero, and $\eta$ is in the range (7), then (4) has no solution such that $x(M)$ is complete with respect to $G^\sharp$. So we can also say that when $\eta$ is in the range (7), the only chance for an entire solution of (4) for which $x(M)$ is complete with respect to $G^\sharp$ and is not an elliptic paraboloid is when $f$ does not satisfy (8). Of course if $\eta$ is not in the range (7), we know nothing about solutions $\rho$ of (4) from Theorem 1.1.

5. In Appendix 1, we modify one step in our argument along the same lines as [7], to prove a similar result to Theorem 1.1, with a more restrictive range of $\eta$, but where the condition on $f$ is independent of $\eta$.
6. In Appendix 2, we detail how similar computations can be used to show that the only solutions to a certain class of second order nonlinear equations on manifolds are identically constant, a generalisation of the result for entire harmonic functions in [14] and [10].

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## 2 Lower bound on Laplacian of test function

As in [6], we bound from above and below the Laplacian of the function

$$\Phi = \frac{\|\nabla \rho\|_G^2}{\rho}.$$

Our techniques are similar to those of Li and Jia, which are similar to those of Yau [10,14] for obtaining a gradient estimate for positive harmonic functions on complete Riemannian manifolds.

In this section, we establish the lower bound on $\frac{\Delta\Phi}{\Phi}$.

**Lemma 2.1** *On the hypersurface $x(M)$ given as the graph of $z$, where $z$ satisfies Eq. 4, the test function $\Phi$ satisfies*

$$\frac{\Delta\Phi}{\Phi} \geq \frac{n}{2(n-1)}\frac{\sum_i \Phi_{,i}^2}{\Phi^2} + \left(\frac{n-2}{n-1}\right)\left\{2\eta - \frac{(n+1)}{2}\right\}\frac{\sum_i \rho_{,i}\Phi_{,i}}{\rho\,\Phi} + C_0\frac{\rho_{,1}^2}{\rho^2} + \frac{2}{n-1}\frac{f^2}{\rho\,\Phi} + \frac{2f}{n-1}\frac{\sum_i \rho_{,i}\Phi_{,i}}{\rho\,\Phi^2} + \left(\frac{2n-4\eta}{n-1}\right)\frac{f}{\rho} - 2f', \quad (9)$$

*where*

$$C_0(\eta, n) := \frac{2}{(n-1)}\eta^2 - \frac{2n}{(n-1)}\eta + \left[2 - \frac{(n^2-2)}{2(n-1)} - \frac{(n-2)^2(n-1)}{8n}\right]. \quad (10)$$

*Here ',' denotes covariant differentiation with respect to G. We sum over repeated indices.*

*Proof* Let $p \in x(M)$ and choose an orthonormal frame field about $p$. We compute

$$\Phi_{,i} = -\frac{\rho_{,i}}{\rho^2}\sum_j \rho_{,j}^2 + \frac{2}{\rho}\sum_j \rho_{,j}\rho_{,ji} \tag{11}$$

and

$$\Phi_{,ii} = \frac{2\rho_{,i}^2}{\rho^3}\sum_j \rho_{,j}^2 - \frac{\rho_{,ii}}{\rho^2}\sum_j \rho_{,j}^2 - \frac{4\rho_{,i}}{\rho^2}\sum_j \rho_{,j}\rho_{,ji} + \frac{2}{\rho}\sum_j \rho_{,ji}^2 + \frac{2}{\rho}\sum_j \rho_{,j}\rho_{,jii}.$$

Summing over $i$ we have

$$\Delta\Phi = \frac{2}{\rho^3}\|\nabla\rho\|_G^4 - \frac{1}{\rho^2}\Delta\rho\,\|\nabla\rho\|_G^2 - \frac{4}{\rho^2}\sum_{i,j}\rho_{,i}\rho_{,j}\rho_{,ji} + \frac{2}{\rho}\sum_{i,j}\rho_{,ji}^2 + \frac{2}{\rho}\sum_{i,j}\rho_{,j}\rho_{,jii}.$$

In view of (4), this becomes

$$\Delta\Phi = (2-\eta)\frac{\|\nabla\rho\|_G^4}{\rho^3} + f\frac{\|\nabla\rho\|_G^2}{\rho^2} - \frac{4}{\rho^2}\sum_{i,j}\rho_{,i}\rho_{,j}\rho_{,ji} + \frac{2}{\rho}\sum_{i,j}\rho_{,ji}^2 + \frac{2}{\rho}\sum_{i,j}\rho_{,j}\rho_{,jii}.$$

In the case where $\Phi(p) = 0$ we have at $p$,

$$\Delta\Phi = \frac{2}{\rho}\sum_{i,j}\rho_{,ji}^2. \tag{12}$$

Now assume $\Phi(p) \neq 0$. Choose an orthonormal frame field such that, at $p$,

$$\rho_{,1} = \|\nabla\rho\|_G\,,\ \rho_{,i} = 0 \text{ for all } i > 1.$$

Then

$$\Delta\Phi = (2-\eta)\frac{\rho_{,1}^4}{\rho^3} + f\frac{\rho_{,1}^2}{\rho^2} - \frac{4}{\rho^2}\rho_{,1}^2\rho_{,11} + \frac{2}{\rho}\sum_{i,j}\rho_{,ji}^2 + \frac{2}{\rho}\rho_{,1}\sum_i \rho_{,1ii}. \tag{13}$$

Interchanging covariant derivatives, we have

$$\nabla_i\nabla_i\nabla_j\rho = \nabla_i\nabla_j\nabla_i\rho = \nabla_j\nabla_i\nabla_i\rho + R_{iji}{}^k\nabla_k\rho,$$

so again using (4),

$$\sum_i \rho_{,1ii} = \nabla_1\Delta\rho + \sum_i R_{i1i}{}^1\rho_{,1} = -\frac{\eta}{\rho^2}\rho_{,1}^3 + \frac{2\eta}{\rho}\rho_{,1}\rho_{,11} - f'\rho_{,1} + R_{11}\rho_{,1}, \tag{14}$$

where $f'$ is the derivative of $f$ with respect to its argument, $\rho$.

As computed by Li and Jia (see [7]), in the Blaschke metric $G$ on our hypersurface,

$$2R_{11} \geq -(n-2)\frac{\rho_{,11}}{\rho} - \frac{(n-2)^2(n-1)}{8n}\frac{\rho_{,1}^2}{\rho^2} + nL_1.$$

In view of (5), this becomes

$$2R_{11} \geq -(n-2)\frac{\rho_{,11}}{\rho} - \frac{(n-2)^2(n-1)}{8n}\frac{\rho_{,1}^2}{\rho^2} - \eta\frac{\rho_{,1}^2}{\rho^2} + \frac{f}{\rho}$$

and hence from (14),

$$2\sum_i \rho_{,1ii} \geq [4\eta - (n-2)]\,\frac{\rho_{,1}\rho_{,11}}{\rho} - \left[3\eta + \frac{(n-2)^2\,(n-1)}{8n}\right]\frac{\rho_{,1}^3}{\rho^2} - 2f'\rho_{,1} + \frac{f}{\rho}\rho_{,1},$$

so substitution back into (13) yields

$$\Delta\Phi \geq [4\eta - (n+2)]\,\frac{\rho_{,1}^2\rho_{,11}}{\rho^2} + \frac{2}{\rho}\sum_{i,j}\rho_{,ji}^2 + \left[2 - 4\eta - \frac{(n-2)^2\,(n-1)}{8n}\right]\frac{\rho_{,1}^4}{\rho^3} - \frac{2f'\rho_{,1}^2}{\rho} + \frac{2f\rho_{,1}^2}{\rho^2} \tag{15}$$

We also estimate using the Cauchy–Schwarz inequality,

$$\sum_{i,j}\rho_{,ji}^2 = \rho_{,11}^2 + \sum_{i>1}\rho_{,ii}^2 + 2\sum_{i<j}\rho_{,ij}^2 \geq \rho_{,11}^2 + \sum_{i>1}\rho_{,ii}^2 + 2\sum_{i>1}\rho_{,1i}^2$$
$$\geq \rho_{,11}^2 + \frac{1}{n-1}\left(\sum_{i>1}\rho_{,ii}\right)^2 + 2\sum_{i>1}\rho_{,1i}^2. \tag{16}$$

In view of (4),

$$\sum_{i>1}\rho_{,ii} = \eta\frac{\rho_{,1}^2}{\rho} - f - \rho_{,11}$$

so

$$\left(\sum_{i>1}\rho_{,ii}\right)^2 = \eta^2\frac{\rho_{,1}^4}{\rho^2} + f^2 + \rho_{,11}^2 - 2\eta f\frac{\rho_{,1}^2}{\rho} - 2\eta\frac{\rho_{,11}\rho_{,1}^2}{\rho} + 2f\rho_{,11}.$$

Therefore, (16) becomes

$$\sum_{i,j}\rho_{,ji}^2 \geq \frac{n}{n-1}\rho_{,11}^2 + 2\sum_{i>1}\rho_{,1i}^2 - \frac{2\eta}{n-1}\frac{\rho_{,11}\rho_{,1}^2}{\rho} + \frac{2f\rho_{,11}}{n-1} + \frac{\eta^2}{n-1}\frac{\rho_{,1}^4}{\rho^2} + \frac{f^2}{n-1} - \frac{2\eta}{n-1}\frac{f\rho_{,1}^2}{\rho} \tag{17}$$

and we have from (15)

$$\Delta\Phi \geq \frac{2n}{n-1}\frac{\rho_{,11}^2}{\rho} + \frac{4}{\rho}\sum_{i>1}\rho_{,1i}^2 + \left\{4\left(\frac{n-2}{n-1}\right)\eta - (n+2)\right\}\frac{\rho_{,1}^2\rho_{,11}}{\rho^2} + \left\{\frac{2}{(n-1)}\eta^2 - 4\eta + 2 - \frac{(n-2)^2\,(n-1)}{8n}\right\}\frac{\rho_{,1}^4}{\rho^3} + \frac{4}{n-1}\frac{f\rho_{,11}}{\rho} + \frac{2}{n-1}\frac{f^2}{\rho} + \left[2 - \frac{4\eta}{n-1}\right]\frac{f\rho_{,1}^2}{\rho^2} - 2\frac{f'\rho_{,1}^2}{\rho}. \tag{18}$$

Similarly as in [6] and [7], we now compute using (11) that

$$\sum_i \Phi_{,i}^2 = \frac{\rho_{,1}^6}{\rho^4} - \frac{4\rho_{,1}^4 \rho_{,11}}{\rho^3} + \frac{4\rho_{,1}^2}{\rho^2} \sum_{i=1}^n \rho_{,1i}^2 \tag{19}$$

so

$$\rho_{,11}^2 = \frac{\rho}{4\Phi} \left\{ \sum_i \Phi_{,i}^2 + \frac{4\rho_{,1}^4 \rho_{,11}}{\rho^3} - \frac{4\rho_{,1}^2}{\rho^2} \sum_{i>1} \rho_{,1i}^2 - \frac{\rho_{,1}^6}{\rho^4} \right\}, \tag{20}$$

and

$$\sum_i \rho_{,i} \Phi_{,i} = -\frac{\rho_{,1}^4}{\rho^2} + \frac{2\rho_{,1}^2 \rho_{,11}}{\rho}$$

which means

$$\rho_{,1}^2 \rho_{,11} = \frac{\rho}{2} \left\{ \sum_i \rho_{,i} \Phi_{,i} + \frac{\rho_{,1}^4}{\rho^2} \right\}. \tag{21}$$

Substituting (20) into (18) yields

$$\begin{aligned} \Delta\Phi \geq & \frac{n}{2(n-1)} \frac{\sum_i \Phi_{,i}^2}{\Phi} + \frac{2(n-2)}{(n-1)\rho} \sum_{i>1} \rho_{,1i}^2 + \left( \frac{n-2}{n-1} \right) \{4\eta - (n+1)\} \frac{\rho_{,1}^2 \rho_{,11}}{\rho^2} \\ & + \left\{ \frac{2}{(n-1)} \eta^2 - 4\eta + \left[ 2 - \frac{n}{2(n-1)} - \frac{(n-2)^2 (n-1)}{8n} \right] \right\} \frac{\rho_{,1}^4}{\rho^3} \\ & + \frac{4}{n-1} \frac{f\rho_{,11}}{\rho} + \frac{2}{n-1} \frac{f^2}{\rho} + \left[ 2 - \frac{4\eta}{n-1} \right] \frac{f\rho_{,1}^2}{\rho^2} - 2\frac{f'\rho_{,1}^2}{\rho}. \end{aligned}$$

Now neglecting the positive $\sum_{i>1} \rho_{,1i}^2$ term, substituting in (21) and dividing through by $\Phi$ we obtain (9). Note that $C_0 > 0$ when

$$\eta > \frac{1}{2} \left\{ n + \frac{(n+2)(n-1)}{2\sqrt{n}} \right\} \text{ or } \eta < \frac{1}{2} \left\{ n - \frac{(n+2)(n-1)}{2\sqrt{n}} \right\}.$$

□

## 3 Upper bound on Laplacian of test function

Next we obtain an upper estimate on $\frac{\Delta\Phi}{\Phi}$. We use the argument of Li and Jia [7], which is a modification of that in [6] and [13], and a related argument of Schoen and Yau [10].

Let $p_0 \in x(M)$. By adding a linear function to $z$ and taking a coordinate transformation, we may assume $p_0$ has coordinates $(0, \dots, 0)$ and

$$z(p_0) = 0, \, z_i(p_0) = 0 \text{ for all } i \text{ and } z_{ij}(p_0) = \delta_{ij}.$$

Let $r(p) = d^\sharp(p_0, p)$, the geodesic distance function from $p_0$ with respect to the metric $G^\sharp$. For any $a > 0$, let $B_a(p_0) = \{p \in x(M) : r(p) \leq a\}$. Consider the function $J : B_a(p_0) \to \mathbb{R}$ defined by

$$J(r) = \left(a^2 - r^2\right)^2 \Phi.$$

Clearly $J$ is nonnegative on $B_a(p_0)$ and attains its maximum at some interior point $p^*$. We may assume $r^2$ is twice differentiable in a neighbourhood of $p^*$ and $\|\nabla\rho\| > 0$ at $p^*$ (otherwise $\|\nabla\rho\| \equiv 0$).

Since $J$ has a local maximum at $p^*$, we have at this point,

$$0 = J_{,i} = -2\left(a^2 - r^2\right)\left(r^2\right)_{,i}\Phi + \left(a^2 - r^2\right)^2\Phi_{,i} \tag{22}$$

and

$$\begin{aligned} 0 \geq \Delta J = 2\Phi\left\|\nabla r^2\right\|_G^2 - 2\left(a^2 - r^2\right)\Phi\Delta r^2 - 4\left(a^2 - r^2\right)\left\langle\nabla r^2, \nabla\Phi\right\rangle_G \\ + \left(a^2 - r^2\right)^2\Delta\Phi. \end{aligned}$$

Dividing through by $\left(a^2 - r^2\right)^2\Phi$ we have at $p^*$,

$$\frac{\Delta\Phi}{\Phi} \leq \frac{2\Delta r^2}{\left(a^2 - r^2\right)} - \frac{2\left\|\nabla r^2\right\|_G^2}{\left(a^2 - r^2\right)^2} + \frac{4}{\left(a^2 - r^2\right)\Phi}\left\langle\nabla r^2, \nabla\Phi\right\rangle_G. \tag{23}$$

## 4 $x(M)$ is an elliptic paraboloid

In this section, we combine our upper and lower estimates on $\frac{\Delta\Phi}{\Phi}$ on the ball $B_a(p_0)$ and let $a \to \infty$ to conclude that $\Phi$ is identically constant and hence $x(M)$ is an elliptic paraboloid.

Combining (23) with (9) we have at $p^*$,

$$\begin{aligned} C_0\frac{\|\nabla\rho\|_G^2}{\rho^2} \leq \frac{2\Delta r^2}{\left(a^2 - r^2\right)} - \frac{2\left\|\nabla r^2\right\|_G^2}{\left(a^2 - r^2\right)^2} - \frac{n}{2(n-1)}\frac{\|\nabla\Phi\|_G^2}{\Phi^2} \\ + \frac{4}{\left(a^2 - r^2\right)}\frac{\left\langle\nabla r^2, \nabla\Phi\right\rangle_G}{\Phi} - \left(\frac{n-2}{n-1}\right)\left\{2\eta - \frac{n+1}{2}\right\}\frac{\langle\nabla\rho, \nabla\Phi\rangle_G}{\rho\,\Phi} \\ + \frac{2}{n-1}\frac{f^2}{\rho\,\Phi} - \frac{2f}{n-1}\frac{\langle\nabla\rho, \nabla\Phi\rangle_G}{\rho\,\Phi} - \left(\frac{2n-4\eta}{n-1}\right)\frac{f}{\rho} + 2f'. \end{aligned} \tag{24}$$

From (22) we have at $p^*$,

$$\frac{\|\nabla\Phi\|_G^2}{\Phi^2} = \frac{4\left\|\nabla r^2\right\|_G^2}{\left(a^2 - r^2\right)^2},$$

$$\frac{\langle\nabla\rho, \nabla\Phi\rangle_G}{\Phi} = \frac{2\left\langle\nabla r^2, \nabla\rho\right\rangle_G}{\left(a^2 - r^2\right)}$$

and

$$\frac{\left\langle\nabla r^2, \nabla\Phi\right\rangle_G}{\Phi} = \frac{2\left\|\nabla r^2\right\|_G^2}{\left(a^2 - r^2\right)}.$$

Substituting these into (24) yields, at $p^*$,

$$\begin{aligned}C_0\frac{\|\nabla\rho\|_G^2}{\rho^2} \leq \frac{2\Delta r^2}{(a^2-r^2)} &+ \left[6-\frac{2n}{n-1}\right]\frac{\|\nabla r^2\|_G^2}{(a^2-r^2)^2}\\ &-\left(\frac{n-2}{n-1}\right)[4\eta-(n+1)]\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{\rho\,(a^2-r^2)}-\frac{4}{(n-1)}\frac{f}{\rho\,\Phi}\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{(a^2-r^2)}\\ &-\frac{2}{(n-1)}\frac{f^2}{\rho\,\Phi}-\left(\frac{2n-4\eta}{n-1}\right)\frac{f}{\rho}+2f'. \end{aligned} \tag{25}$$

We convert the Laplacian of $r^2$ into the $G^\sharp$ metric. Using (2),

$$\Delta r = \frac{1}{\rho}\Delta^\sharp r + \frac{(n-2)}{2\rho}\langle\nabla\rho,\nabla r\rangle_G$$

and therefore

$$\Delta r^2 = \frac{2r}{\rho}\Delta^\sharp r + \frac{(n-2)}{2\rho}\left\langle\nabla\rho,\nabla r^2\right\rangle_G + 2\,\|\nabla r\|_G^2.$$

Hence the right hand side of (25) is less than or equal to

$$\begin{aligned}&\frac{4r\Delta^\sharp r}{\rho\,(a^2-r^2)}+\frac{4\,\|\nabla r\|_G^2}{(a^2-r^2)}+\left[6-\frac{2n}{n-1}\right]\frac{\|\nabla r^2\|_G^2}{(a^2-r^2)^2}\\ &-2\left(\frac{n-2}{n-1}\right)(2\eta-n)\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{\rho\,(a^2-r^2)}-\frac{4}{(n-1)}\frac{f}{\rho\,\Phi}\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{(a^2-r^2)}\\ &-\frac{2}{(n-1)}\frac{f^2}{\rho\,\Phi}-\left(\frac{2n-4\eta}{n-1}\right)\frac{f}{\rho}+2f'. \end{aligned} \tag{26}$$

Finally we estimate

$$\begin{aligned}&-2\left(\frac{n-2}{n-1}\right)(2\eta-n)\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{\rho\,(a^2-r^2)}\\ &\quad\leq \frac{1}{a}\frac{\|\nabla\rho\|_G^2}{\rho^2}+a\left(\frac{n-2}{n-1}\right)^2(2\eta-n)^2\frac{\|\nabla r^2\|_G^2}{(a^2-r^2)^2}\end{aligned}$$

and, recalling the definition of $\Phi$,

$$-\frac{4}{(n-1)}\frac{f}{\rho\,\Phi}\frac{\langle\nabla r^2,\nabla\rho\rangle_G}{(a^2-r^2)} \leq \frac{2}{(n-1)}\frac{f^2}{\rho\,\Phi}+\frac{2}{(n-1)}\frac{\|\nabla r^2\|_G^2}{(a^2-r^2)^2}$$

Substituting these into (26) we find from (25) that at $p^*$,

$$\begin{aligned}\left[C_0-\frac{1}{a}\right]\frac{\|\nabla\rho\|_G^2}{\rho^2} \leq \frac{4r\Delta^\sharp r}{\rho\,h\,(a^2-r^2)}+\frac{4\,\|\nabla r\|_G^2}{(a^2-r^2)}+\left[2f'-\left(\frac{2n-4\eta}{n-1}\right)\frac{f}{\rho}\right]\\ +\left[4+a\left(\frac{n-2}{n-1}\right)^2(2\eta-n)^2\right]\frac{\|\nabla r^2\|_G^2}{(a^2-r^2)^2}. \end{aligned} \tag{27}$$

If we require $f$ to satisfy the ordinary differential inequality

$$f' - \left(\frac{n-2\eta}{n-1}\right)\frac{f}{\rho} \le 0$$

then from (27) we have at $p^*$

$$\left[C_0 - \frac{1}{a}\right]\frac{\|\nabla\rho\|_G^2}{\rho^2} \le \frac{4r\Delta^\sharp r}{\rho\left(a^2-r^2\right)} + \frac{4\|\nabla r\|_G^2}{\left(a^2-r^2\right)} + \left[4 + a\left(\frac{n-2}{n-1}\right)^2(2\eta-n)^2\right]\frac{r^2\|\nabla r\|_G^2}{\left(a^2-r^2\right)^2}.$$

Multiplying through by $\left(a^2-r^2\right)^2$ we get at $p^*$

$$\left[C_0 - \frac{1}{a}\right]\frac{J}{\rho} \le 4\left(a^2-r^2\right)\left(\frac{r\Delta^\sharp r}{\rho} + \|\nabla r\|_G^2\right) + \left[4 + a\left(\frac{n-2}{n-1}\right)^2(2\eta-n)^2\right]r^2\|\nabla r\|_G^2.$$

Converting the norms on the right hand side into the $G^\sharp$ metric and using $\|\nabla r\|_{G^\sharp} = 1$, this becomes, at $p^*$,

$$\left[C_0 - \frac{1}{a}\right]J \le \left(a^2-r^2\right)\left(r\Delta^\sharp r + 1\right) + \left[4 + a\left(\frac{n-2}{n-1}\right)^2(2\eta-n)^2\right]r^2. \tag{28}$$

Let $a^* = r(p^*)$. If $p^* \ne p_0$ then $a^* > 0$. We have the following Lemma as in [7].

**Lemma 4.1** *For $p \in B_{a^*}(p_0)$,*

(i) $\Phi(p) \le \Phi(p^*)$,
(ii) *The Ricci curvature of $x(M)$ in the $G^\sharp$ metric satisfies*

$$R^\sharp_{ij}(p) \ge -\frac{(n+2)^2}{16}\Phi\left(p^*\right)z_{ij}(p),$$

(iii) $r\Delta^\sharp r \le (n-1)\left(1 + \frac{(n+2)}{4}\sqrt{\Phi(p^*)}r\right)$.

*Proof*

(i) Under conditions (7) and (8), we have from (9)

$$\Delta\Phi \ge \sum_i\left\{\frac{n}{2(n-1)}\Phi_{,i} + \left(\frac{n-2}{n-1}\right)\frac{\rho_{,i}}{\rho} + \frac{2f}{n-1}\frac{\rho_{,i}}{\rho\,\Phi}\right\}\Phi_{,i}, \tag{29}$$

so by the maximum principle,

$$\max_{B_{a^*}(p_0)}\Phi = \max_{\partial B_{a^*}(p_0)}\Phi. \tag{30}$$

Observe also that

$$J\left(p^*\right) = \left(a^2-\left(a^*\right)^2\right)^2\Phi\left(p^*\right) \ge \max_{p\in\partial B_{a^*}(p_0)}\left(a^2 - r(p)^2\right)^2\Phi(p) = \left(a^2-\left(a^*\right)^2\right)^2\max_{p\in\partial B_{a^*}(p_0)}\Phi(p)$$